Math 247A Lecture 14 Notes

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1 The Sobolev Embedding Theorem

1.1 Fourier transforms of tempered distributions

Fix $0 < \alpha < d$, and consider

$$\int e^{-\pi t|x|^2} t^{(d-\alpha)/2} \frac{dt}{t}.$$

If we let $u = \pi |x|^2 t$, then this equals

$$\begin{aligned} \int_0^\infty e^{-u} \left(\frac{u}{\pi |x|^2} \right)^{(d-\alpha)/2} \frac{du}{u} &= \pi^{(d-\alpha)/2} \frac{1}{|x|^{d-\alpha}} \int_0^\infty e^{-u} u^{(d-\alpha)/2} \frac{du}{u} \\ &= \pi^{-(d-\alpha)/2} \Gamma\left(\frac{d-\alpha}{2} \right) \frac{1}{|x|^{d-\alpha}}. \end{aligned}$$

We regard $\pi^{-(d-\alpha)/2}\Gamma\left(\frac{d-\alpha}{2}\right)\frac{1}{|x|^{d-\alpha}}$ as a **tempered distribution**, that is an element of $\mathcal{S}'(\mathbb{R}^d)$. These are linear functionals on $\mathcal{S}(\mathbb{R}^d)$.

For $T \in \mathcal{S}'(\mathbb{R}^d)$givenby a density φ,,

$$T(f) = \int f(x) \varphi(x)\, dx.$$

In our case,

$$\left(\pi^{-(d-\alpha)/2} \Gamma\left(\frac{d-\alpha}{2} \right) \frac{1}{|x|^{d-\alpha}} \right)(f) = \pi^{-(d-\alpha)/2} \Gamma\left(\frac{d-\alpha}{2} \right) \int \frac{f(x)}{|x|^{d-\alpha}}\, dx.$$

Since f is a Schwarz function, this integrand has the right decay at ∞. We have

$$\begin{aligned} \left| \int \frac{f(x)}{|x|^{d-\alpha}}\, dx \right| &\leq \left| \int_{|x| \leq R} \frac{f(x)}{|x|^{d-\alpha}}\, dx \right| + \left| \int_{|x|>R} \frac{f(x)}{|x|^{d-\alpha}}\, dx \right| \\ &\lesssim \|f\|_\infty \int \frac{1}{|x|^{d-\alpha}}\, dx + \||x|^d f\|_{L^\infty} R^{-(d-\alpha)}. \end{aligned}$$

Definition 1.1. For $T \in \mathcal{S}'(\mathbb{R}^d)$, we define its **Fourier transform** by

$$\widehat{T}(f) = T(\widehat{f}), \qquad f \in \mathcal{S}(\mathbb{R}^d).$$

Let's compute the Fourier transform of $\pi^{-(d-\alpha)/2}\Gamma\left(\frac{d-\alpha}{2}\right)\frac{1}{|x|^{d-\alpha}}$:

$$\begin{aligned}
\left(\pi^{-(d-\alpha)/2}\Gamma\left(\frac{d-\alpha}{2}\right)\frac{1}{|x|^{d-\alpha}}\right)^{\wedge}(f) &= \pi^{-(d-\alpha)/2}\Gamma\left(\frac{d-\alpha}{2}\right)\int \frac{\widehat{f}(x)}{|x|^{d-\alpha}}\,dx \\
&= \pi^{-(d-\alpha)/2}\Gamma\left(\frac{d-\alpha}{2}\right)\iint \frac{e^{-2\pi i x\cdot\xi}}{|x|^{d-\alpha}} f(\xi)\,dx\,d\xi \\
&= \int_0^\infty \iint e^{-\pi t|x|^2} t^{(d-\alpha)/2} e^{-2\pi i x\cdot \xi} f(\xi)\,dx\,d\xi\,\frac{dt}{t}
\end{aligned}$$

We already know the Fourier transform of a Gaussian.

$$\begin{aligned}
&= \int_0^\infty \int \pi^{d/2}(\pi t)^{-d/2} e^{-\pi/t\cdot|\xi|^2} t^{(d-\alpha)/2} f(\xi)\,d\xi\,\frac{dt}{t} \\
&= \int_0^\infty \int e^{-\pi/t\cdot|\xi|^2} t^{-\alpha/2} f(\xi)\,d\xi\,\frac{dt}{t}
\end{aligned}$$

Make the change of variables $u = \pi|\xi|^2/t$.

$$\begin{aligned}
&= \int\int_0^\infty f(\xi)e^{-u}\left(\frac{u}{\pi|\xi|^2}\right)^{\alpha/2}\frac{du}{u}\,d\xi \\
&= \int \pi^{-\alpha/2}\frac{1}{|\xi|^\alpha}\int_0^\infty e^{-u}u^{\alpha/2}\frac{du}{u} f(\xi)\,d\xi \\
&= \left[\pi^{-\alpha/2}\Gamma(\alpha/2)\frac{1}{|\xi|^\alpha}\right](f)
\end{aligned}$$

Remark 1.1. Take $d = 3$ and $\alpha = 2$:

$$\left(\pi^{-1/2}\Gamma(1/2)\frac{1}{|x|}\right)^{\wedge} = \pi^{-1}\Gamma(1)\frac{1}{|\xi|^2}.$$

That is,

$$\left(\frac{1}{2\pi|x|}\right)^{\wedge} = \frac{1}{4\pi^2|\xi|^2}.$$

This allows us to solve Poisson's equation: $-\Delta u = f$. If we take the Fourier transform, this is

$$4\pi^2|\xi|^2\widehat{u}(\xi) = \widehat{f}(\xi),$$

so

$$\widehat{u}(\xi) = \frac{1}{4\pi^2|\xi|^2}\widehat{f}(\xi).$$

Taking the inverse Fourier transform, we get

$$u = \frac{1}{4\pi|x|} * f.$$

To make everything rigorous, use

- If $T \in \mathcal{S}'(\mathbb{R}^d)$ and $f \in \mathcal{S}(\mathbb{R}^d)$, then $T * f \in \mathcal{S}'(\mathbb{R}^d)$ is given by $(T * f)(g) = T(f_R * g)$, where $f_R(x) = f(-x)$:

$$\begin{aligned}(T * f)(g) &= \int (\varphi * f)(x) g(x)\, dx \\ &= \iint \varphi(y) f(x-y) g(x)\, dx\, dy \\ &= T(f_R * g).\end{aligned}$$

- If $T \in \mathcal{S}'(\mathbb{R}^d)$ and $f \in \mathcal{S}(\mathbb{R}^d)$, then $\widehat{T * f} = \widehat{T}\widehat{f}$.

1.2 Sobolev embedding

Definition 1.2. Fix $s > -d$ and $f \in \mathcal{S}(\mathbb{R}^d)$. Then $|\nabla|^s f \in \mathcal{S}'(\mathbb{R}^d)$ is defined by its action on the Fourier side:

$$(|\nabla|^s f)^\wedge(\xi) = (2\pi|\xi|)^s \widehat{f}(\xi).$$

Theorem 1.1 (Sobolev embedding)**.** *For $f \in \mathcal{S}(\mathbb{R}^d)$ and $0 < s < d$, we have*

$$\|f\|_q \lesssim \||\nabla|^s f\|_p$$

whenever $\frac{1}{p} = \frac{1}{q} + \frac{s}{d}$. The implicit constant is independent of f.

What does this say? It says that if s derivatives of f live in L^p, then the function must be more regular/smooth (it lives is a higher L^p space).

Proof. By duality, $\|f\|_q = \sup_{\|g\|_{L^{q'}}=1} \langle f, g\rangle$. The idea is that by Plancherel,

$$\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle = \langle (2\pi|\xi|)^s \widehat{f}, (2\pi|\xi|^{-s}\widehat{g}(\xi)\rangle,$$

where the first argument is in $\mathcal{S}'$.

We claim that $\mathcal{F} = \{g \in \mathcal{S}'(\mathbb{R}^d) : \widehat{g} \text{ vanishes on a neighborhood of } 0\}$ is dense in $L^{q'}$. It suffices to show that $\mathcal{F}$ is dense in $\mathcal{S}(\mathbb{R}^d)$ in the topology of $L^{q'}$. Fix $g_0 \in \mathcal{S}(\mathbb{R}^d)$. Fix $\varepsilon > 0$ and $\varphi \in C_c^\infty(B(0,2))$ with $\varphi \equiv 1$ on $B(0,1)$. Define $\widehat{g_\varepsilon(\xi)} = \widehat{g_0(\xi)}(1 - \varphi(\xi/\varepsilon)) \in \mathcal{S}$. Then $g_\varepsilon \in \mathcal{F}$. Then $\widehat{g}_0 - \widehat{g}_\varepsilon = \widehat{g}_0\varphi(\cdot/\varepsilon)$, so $g_0 - g_\varepsilon = g_0 * \varepsilon^d \varphi^\vee(\varepsilon\,\cdot)$.

$$\|g_0 - g_\varepsilon\|_{q'} \lesssim \|g_0\|_1 \|\varepsilon^d \varphi^\vee(\varepsilon\,\cdot)\|_{q'}$$

$$\lesssim \|g_0\|_1 \varepsilon^{d-d/q'}$$
$$\xrightarrow{\varepsilon\to 0} 0.$$

Then

$$\begin{aligned}
\|f\|_q &= \sup_{g\in\mathcal{F}:\|g\|_{q'}=1} \langle f, g\rangle \\
&= \sup_{g\in\mathcal{F}:\|g\|_{q'}=1} \langle \underbrace{(2\pi|\xi)^s \widehat{f}}_{\in\mathcal{S}'}, \underbrace{(2\pi|\xi|)^{-s}\widehat{g}}_{\in\mathcal{S}}\rangle \\
&= \sup_{g\in\mathcal{F}:\|g\|_{q'}=1} \langle \underbrace{|\nabla|^s f}_{\in\mathcal{S}'}, \underbrace{|\nabla^{-s} g}_{\in\mathcal{S}}\rangle \\
&\lesssim \sup_{g\in\mathcal{F}:\|g\|_{q'}=1} \||\nabla|^s f\|_p \cdot \||\nabla|^{-s} g\|_{p'}
\end{aligned}$$

We have

$$|\nabla|^{-s} g = [(2\pi|\xi|)-s\widehat{g}]^\vee \sim \frac{1}{|x|^{d-s}} * g,$$

so

$$\||\nabla|^{-s} g\|_{p'} \sim \left\| \frac{1}{|x|^{d-s} * g} \right\|_{p'} \lesssim \|g\|_{q'}$$

by Hardy-Littlewood-Sobolev, provided $1 + \frac{1}{p'} = \frac{1}{q'} + \frac{d-s}{d}$. We can rewrite this condition as $\frac{1}{p} = \frac{1}{q} = \frac{s}{d}$. □